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On $L_p(l_q)$-Spaces of Entire Analytic Functions of Exponential Type: Complex Interpolation and Fourier Multipliers. The Case $0 < p < \infty, 0 < q < \infty$

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1. INTRODUCTION

The real interpolation method of Lions and Peetre for pairs of Banach spaces can be extended easily to pairs of quasi-Banach spaces. [A space A is said to be a quasi-Banach space if it has all the properties of a Banach space with the exception of the triangle inequality which is replaced by the property: There exists a positive constant c such that, for all $a \in A$ and all $b \in A$,

$$\| a + b \| \leqslant c(\| a \| + \| b \|).]$$

In contrast to this situation, the complex interpolation method of Calderón, Lions, and Krejn depends heavily on the fact that the underlying spaces are complex Banach spaces. As far as interpolation methods are concerned we refer to [1, 8]. Recently, Calderón and Torchinsky described a complex interpolation method for spaces of Hardy type, which are quasi-Banach spaces; cf. [3, II.3]. As a particular case, one obtains complex interpolation of the usual Hardy spaces in the sense of Fefferman and Stein; cf. [4] (real-variable characterization). In his unfortunately unpublished lecture notes [6] Peetre proved a Littlewood–Paley theorem for the Hardy spaces in the sense of Fefferman and Stein. A statement of this result and a new proof may be found in [9, 3.2.1]. Using that assertion, the Calderón–Torchinsky result mentioned above can be essentially reformulated as the complex interpolation of the two spaces $L^A_{p_0}(l_2)$ and $L^A_{p_1}(l_2)$ of entire analytic functions of exponential type, where $0 < p_0 < \infty$ and $0 < p_1 < \infty$. A precise definition of the spaces $L_p{}^A(l_q)$ with $0 < p < \infty$ and $0 < q < \infty$ is given in Section 2, Eq. (2). The aim of this paper is two fold. First (Section 2, Theorems 1 and 2), we prove a complex interpolation theorem for two spaces $L^A_{p_0}(l_{q_0})$ and $L^A_{p_1}(l_{q_1})$, where $0 < p_0 < \infty$, $0 < q_0 < \infty$, $0 < p_1 < \infty$, $0 < q_1 < \infty$, and $p_0/q_0 = p_1/q_1$. The additional assumption $p_0/q_0 = p_1/q_1$ depends on our method. We

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conjecture that the results formulated in Theorems 1 and 2 are true without this additional assumption. (A proof of these generalized theorems can perhaps be given by a combination of the method of Calderón and Torchinsky [3, II, pp. 144–151] and our method.) Second (Section 3, Theorem 3), on the basis of Theorem 2 we prove a Fourier multiplier theorem for the spaces $L_p{}^A(l_q)$. The formulation looks somewhat curious, but as we shall see in a later paper [11], it is sharp in some sense.

We hope that the results of this paper are of self-contained interest. On the other hand, in [11] we intend to apply the results of this paper to the theory of the spaces $F^s_{p,q}$ of Hardy–Sobolev type that has been developed in [9]. Occasionally in [9] there are some awkward, unnatural looking restrictions for s if $p < 1$. With the help of the theorems proved in the present paper (in particular Theorem 3) these restrictions can be removed and replaced by natural conditions. Among other things, we shall be concerned in [11] with equivalent quasi-norms in $F^s_{p,q}$ in the sense of [10].

2. Complex Interpolation of $L_p{}^A(l_q)$

The n-dimensional real Euclidean space and its points are denoted by R_n and $x = (x_1, ..., x_n)$, respectively. S is the Schwartz space of all rapidly decreasing infinitely differentiable complex-valued functions on R_n, and S' is the space of all complex-valued tempered distributions. (Because the dimension n is fixed once for all we omit R_n and simply write S instead of $S(R_n)$, etc.) F is the Fourier transform on S', and F^{-1} is the inverse Fourier transform. We recall the famous Paley–Wiener–Schwartz theorem, which states that $f \in S'$ is an entire analytic function of exponential type if Ff has a compact support. If

$$D_k = \{x \mid |x| \leqslant 2^k\}, \qquad k = 0, 1, 2, ..., \tag{1}$$

and if $0 < p < \infty$ and $0 < q < \infty$, then

$$\begin{aligned} L_p{}^A(l_q) = \Big\{ f \mid f = \{f_j\}_{j=0}^{\infty},\ f_k \in S',\ \operatorname{supp} Ff_k \subset D_k \text{ for } k = 0, 1, 2, ..., \\ \|f \mid L_p(l_q)\| = \Big(\int_{R_n} \Big(\sum_{l=0}^{\infty} |f_l(x)|^q \Big)^{p/q} dx \Big)^{1/p} < \infty \Big\}. \end{aligned} \tag{2}$$

If $\operatorname{supp} Ff_k \subset D_k$ in the definition of $L_p{}^A(l_q)$ is replaced by $\operatorname{supp} Ff_k \subset D_{k+1}$, where again $k = 0, 1, 2, ...$, then the corresponding space is denoted by $L_p^{2A}(l_q)$ (the number 2 indicates that the balls D_{k+1} are doubled in size in comparison with D_k).

If $0 < p_0 < \infty$, $0 < p_1 < \infty$, $0 < q_0 < \infty$, and $0 < q_1 < \infty$, then

$F(L_{p_0}^{2A}(l_{q_0}), L_{p_1}^{2A}(l_{q_1}))$ denotes the space of all systems $f = \{f_k(x, z)\}_{k=0}^{\infty}$ of functions with the following properties:

(i) If $k = 0, 1, 2, \ldots$, then $f_k(x, z)$ is defined for $x \in R_n$ and complex z with $0 \leqslant \operatorname{Re} z \leqslant 1$. It is a bounded and continuous funtcion in $R_n \times \{z \mid 0 \leqslant \operatorname{Re} z \leqslant 1\}$. If $x \in R_n$ is fixed, then $f_k(x, z)$ is analytic in $0 < \operatorname{Re} z < 1$.

(ii) If z with $0 \leqslant \operatorname{Re} z \leqslant 1$ is fixed, then $f_k(x, z)$ is an entire analytic function and

$$\operatorname{supp}(Ff_k)(\cdot, z) \subset D_{k-1}, \qquad k = 0, 1, 2, \ldots . \tag{3}$$

(iii) If $\| \cdot \mid L_p(l_q)\|$ has the same meaning as in (2), then

$$\begin{aligned} \| f \mid F(L_{p_0}^{2A}(l_{q_0}), L_{p_1}^{2A}(l_{q_1}))\| \\ = \max_{l=0,1} \sup_{t \in R_1} \| \{f_j(\cdot, l - it)\}_{j=0}^{\infty} \mid L_{p_l}(l_{q_l}) \| < \infty. \end{aligned} \tag{4}$$

The set of all systems $f = \{f_k(x, z)\}_{k=0}^{\infty}$ satisfying only the above properties (i) and (ii) is denoted by M^{2A}.

Remark 1. This is the counterpart of the well-known construction introduced by Calderón in [2] in the case of Banach spaces. A description may also be found in [8, p. 56].

PROPOSITION 1. (i) *If* $0 < p < \infty$ *and* $0 < q < \infty$ *then* $L_p{}^A(l_q)$ *equipped with the quasi-norm* (2) *is a quasi-Banach space* (*Banach space if* $1 \leqslant p$ *and* $1 \leqslant q$).

(ii) *Equation* (4) *is a quasi-norm.* $F(L_{p_0}^{2A}(l_{q_0}), L_{p_1}^{2A}(l_{q_1}))$ *equipped with this quasi-norm is a quasi-Banach space* (*Banach space if* p_0, p_1, q_0, *and* q_1 *are larger than or equal to* 1).

Proof. Step 1. We prove (ii). All the required properties are clear with the exception of (α) If $\| f \mid F(L_{p_0}^{2A}(l_{q_0}), L_{p_1}^{2A}(l_{q_1}))\| = 0$, then $f_k(x, z) = 0$ for $k = 0, 1, 2, \ldots$ and (β) $F(L_{p_0}^{2A}(l_{q_0}), L_{p_1}^{2A}(l_{q_1}))$ is complete. In order to prove (α) we recall the well-known formula

$$\begin{aligned} \log |f_k(x, z)| \leqslant \int_{-\infty}^{\infty} \log |f_k(x, it)| \, \mu_0(\operatorname{Re} z, t) \, dt \\ + \int_{-\infty}^{\infty} \log |f_k(x, 1 + it)| \, \mu_1(\operatorname{Re} z, t) \, dt. \end{aligned} \tag{5}$$

Here $x \in R_n$, $0 < \operatorname{Re} z < 1$ and $k = 0, 1, 2, \ldots$.

Furthermore, $\mu_0(\theta, t)$ and $\mu_1(\theta, t)$ are positive kernels with

$$\frac{1}{1-\theta}\int_{-\infty}^{\infty} \mu_0(\theta, t)\, dt = \frac{1}{\theta}\int_{-\infty}^{\infty} \mu_1(\theta, t)\, dt = 1, \tag{6}$$

cf., e.g., [8, pp. 65, 67]. After multiplication of (5) with $0 < r < \infty$, it follows in the same way as in [8, p. 67] that

$$|f_k(x, z)|^r \leqslant \left(\frac{1}{1-\theta}\int_{-\infty}^{\infty} |f_k(x, it)|^r \mu_0(\theta, t)\, dt\right)^{1-\theta}$$

$$\times \left(\frac{1}{\theta}\int_{-\infty}^{\infty} |f_k(x, 1-it)|^r \mu_1(\theta, t)\, dt\right)^{\theta} \tag{7}$$

with $\theta = \operatorname{Re} z$. Now, ($\alpha$) is a consequence of (7). (The number r will be useful later on.) We prove the completeness of $F(L_{p_0}^{2A}(l_{q_0}), L_{p_1}^{2A}(l_{q_1}))$. If $f^m = \{f_k^m(x, z)\}_{k=0}^{\infty}$, where $m = 1, 2, \ldots$ is a fundamental sequence in $F(L_{p_0}^{2A}(l_{q_0}), L_{p_1}^{2A}(l_{q_1}))$, then $f^m(\cdot, z)$, with $z = it$ or $z = 1 + it$, respectively, t real, converges in the quasi-norm $\| \cdot \mid L_{p_l}(l_{q_l}) \|$ (cf. (2)), where $l = 0, 1$, to the limit element $f = \{f_k(x, z)\}_{k=0}^{\infty}$. In order to prove that the definition of $f_k(x, z)$ can be extended in a natural way to all z with $0 \leqslant \operatorname{Re} z \leqslant 1$, and that the functions so defined have the required additional properties, we use the following Plancherel–Polya–Nikol'skij inequality; cf. [7, p. 37]. If k with $k = 0, 1, 2, \ldots$ is fixed, then there exists a constant c_k such that for all t with $t \in R_1$ and all $m = 1, 2, \ldots$,

$$\sup_{x \in R_n} |f_k^m(x, it)| \leqslant c_k \left(\int_{R_n} |f_k^m(x, it)|^{p_0}\, dx\right)^{1/p_0}. \tag{8}$$

A similar result holds for $f_k^m(x, 1-it)$, where p_0 must be replaced by p_1. Here we used (3) essentially. By (7) with f_k^m instead of f_k, (8) and a limit argument, it follows that $f_k(x, z)$ can be defined in a natural way for all $x \in R_n$ and all z with $0 \leqslant \operatorname{Re} z \leqslant 1$, and that these functions have the required additional properties.

Step 2. The proof of (i) follows by standard arguments from a Plancherel–Polya–Nikol'skij inequality of type (8).

DEFINITION. If $0 < \theta < 1$, $0 < p_0 < \infty$, $0 < p_1 < \infty$, $0 < q_0 \leqslant \infty$, and $0 < q_1 \leqslant \infty$, then $[L_{p_0}^A(l_{q_0}), L_{p_1}^A(l_{q_1})]_\theta$ is the set of all systems $f = \{f_k(x)\}_{k=0}^{\infty}$ of functions for which there exists a system

$$g = \{g_k(x, z)\}_{k=0}^{\infty} \in F(L_{p_0}^{2A}(l_{q_0}), L_{p_1}^{2A}(l_{q_1}))$$

with $f_k(x) = g_k(x, \theta)$ and

$$\operatorname{supp}(Fg_k)(\cdot, \theta) \subset D_k; \qquad k = 0, 1, 2, \ldots. \tag{9}$$

Let

$$\|f \mid [L^A_{p_0}(l_{q_0}), L^A_{p_1}(l_{q_1})]_\theta\| = \inf \| g \mid F(L^{2A}_{p_0}(l_{q_0}), L^{2A}_{p_1}(l_{q_1}))\|, \tag{10}$$

where the infimum is taken over all admissible systems g.

Remark 2. This is the counterpart of the usual definition of complex interpolation in Banach spaces; cf. [2] or [8, p. 58]. If

$$\{g_k(x, z)\}_{k=0}^{\infty} \in F(L^{2A}_{p_0}(l_{q_0}), L^{2A}_{p_1}(l_{q_1})),$$

then the support of $(Fg_k)(x, \theta)$ (where θ is a fixed number) is contained in D_{k+1} by definition; cf. (3). In other words, (9) strengthens this hypothesis.

PROPOSITION 2. *$[L^A_{p_0}(l_{q_0}), L^A_{p_1}(l_{q_1})]_\theta$ is a quasi-Banach space. Furthermore,*

$$\|f \mid [L^A_{p_0}(l_{q_0}), L^A_{p_1}(l_{q_1})]_\theta\|^* = \inf(\sup_{t\in R_1} \|\{g_j(\cdot, it)\} \mid L_{p_0}(l_{q_0})\|)^{1-\theta} (\sup_{t\in R_1} \|\{g_j(\cdot, 1+it)\} \mid L_{p_1}(l_{q_1})\|)^\theta \tag{10*}$$

is an equivalent quasi-norm, where the infinum is taken over all admissible systems g.

Proof. It follows by an inequality of type (8) and standard arguments that $[L^A_{p_0}(l_{q_0}), L^A_{p_1}(l_{q_1})]_\theta$ is a quasi-Banach space. We prove (10*) is an equivalent quasi-norm. Obviously, the quasi-norm in (10*) can be estimated from above by the quasi-norm in (10). In order to prove the reversion we replace the system $g = \{g_j(x, z)\}_{j=0}^{\infty}$ in (10) by $\{a^{z-\theta} g_j(x, z)\}_{j=0}^{\infty}$, where a is a positive number. By (4) we have

$$\|f \mid [L^A_{p_0}(l_{q_0}), L^A_{p_1}(l_{q_1})]_\theta\| \leqslant a^{-\theta} \sup_{t\in R_1} \|\{g_j(x, it)\} \mid L_{p_0}(l_{q_0})\| + a^{1-\theta} \sup_{t\in R_1} \|\{g_j(x, 1+it)\} \mid L_{p_1}(l_{q_1})\|.$$

If one chooses a in an appropriate way (such that the two summands on the right-hand side are equal), and if afterwards one takes the infimum with respect to g, then the right-hand side of the last formula can be estimated from above by the quasi-norm in (10*). Hence, the quasi-norms in (10) and (10*) are equivalent.

THEOREM 1. *If $0 < \theta < 1$, $0 < p_0 < \infty$, $0 < p_1 < \infty$, $0 < q_0 < \infty$, $0 < q_1 < \infty$ and $p_0/q_0 = p_1/q_1$, then*

$$[L^A_{p_0}(l_{q_0}), L^A_{p_1}(l_{q_1})]_\theta = L_p{}^A(l_q), \tag{11}$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

(*equivalent quasi-norms*).

Proof. *Step* 1. (In this step the additional assumption $p_0/q_0 = p_1/q_1$ is not needed.) Let $\{g_k(x, z)\}_{k=0}^{\infty} \in F(L_{p_0}^{2,A}(l_{q_0}), L_{p_1}^{2,A}(l_{q_1}))$ with (9). By (7) with $g_k(x, z)$ instead of $f_k(x, z)$ and $0 < r < \min(p_0, p_1, q_0, q_1)$, it follows by Hölder's inequality that

$$\begin{aligned}
&\| \{g_k(x, \theta)\| \, L_p(l_q) \| \\
&= \| \{ |g_k(x, \theta)|^r\} \, | \, L_{(p/r)}(l_{(q/r)}) \|^{(1/r)} \\
&\leqslant \Big[\int_{R_n} \Big(\frac{1}{1-\theta} \int_{-\infty}^{\infty} \| \{g_k(x, it)\}\|_{l_{q_0}}^r \, \mu_0(\theta, t)\, dt \Big)^{(1-\theta)(p/r)} \\
&\quad \times \Big(\frac{1}{\theta} \int_{-\infty}^{\infty} \| \{g_k(x, 1+it)\} \|_{l_{q_0}}^r \, \mu_1(\theta, t)\, dt \Big)^{\theta(p/r)} dx \Big]^{(1/p)} \\
&\leqslant (\sup_t \| \{g_k(\cdot, it)\| \, L_{p_0}(l_{q_0}) \|^{1-\theta})(\sup_t \| \{g_k(\cdot, 1+it)\| \, L_{p_1}(l_{q_1}) \|^{\theta}) \\
&\leqslant \| g \mid F(L_{p_0}^{2,A}(l_{q_0}), L_{p_1}^{2,A}(l_{q_1})) \| .
\end{aligned}$$

Hence (with topological imbedding)

$$[L_{p_0}^A(l_{q_0}), L_{p_1}^A(l_{q_1})]_\theta \subset L_p{}^A(l_q). \tag{12}$$

Step 2. In order to prove the reversion to (12) we may restrict ourselves to systems $f = \{f_j\}_{j=0}^{\infty} \subset S$ with supp $Ff_k \subset D_k$ for $k = 0, 1, 2, \ldots$, where only a finite number of the functions f_j does not vanish identically. This set is dense in $L_p{}^A(l_q)$ because the set $\{h \mid h \in S, \text{supp } Fh \subset D_k\}$ is dense in

$$\Big\{ H \mid H \in S', \text{supp } FH \subset D_k, \Big(\int_{R_n} |H(x)|^p \, dx \Big)^{1/p} < \infty \Big\};$$

cf. [7, p. 40]. Again $k = 0, 1, 2, \ldots$. Let f be such a system. We introduce the maximal functions

$$f_k^*(x) = \sup_{y \in R_n} \frac{|f_k(x-y)|}{1 + |2^k y|^a}, \qquad a > \frac{n}{\min(p, q)}; \tag{13}$$

cf. [9, 2.2.3, formula (6)]. Again $k = 0, 1, 2, \ldots$. Let $\varphi(x) \in S$ with

$$\varphi_k(x) = 1 \text{ if } |x| \leqslant 1 \qquad \text{and} \qquad \text{supp } \varphi \subset \{y \mid |y| \leqslant 2\}, \tag{14}$$

and $\varphi_k(x) = \varphi(2^{-k}x)$. If z is a complex number with $0 \leqslant \operatorname{Re} z \leqslant 1$ and $j = 0, 1, 2, \ldots$, then we put

$$g_j(x, z) = F^{-1}[\varphi_j F(f_j \, |f_j^*|^{b(z-\theta)} \, \|\{f_k^*\} \mid L_p(l_q)\|^{d(z-\theta)})]. \tag{15}$$

Here b and d are real numbers which will be chosen later on. $\|\cdot \mid L_p(l_q)\|$ has the same meaning as in (2). If f_j does not vanish identically, then $f_j^*(x) > 0$ for all $x \in R_n$. Hence, the construction in (15) is correct, and the funtcion in the brackets ( ) belongs to L_1. Hence, (15) can be rewritten as

$$g_j(x, z) = \|\{f_k^*\} \mid L_p(l_q)\|^{d(z-\theta)} \int_{R_n} (F^{-1}\varphi_j)(y) f_j(x - y) |f_j^*(x - y)|^{b(z-\theta)} \, dy. \tag{16}$$

In order to show that the system $g = \{g_j(x, z)\}_{j=0}^{\infty}$ belongs to $F(L_{p_0}^{2A}(l_{q_0}), L_{p_1}^{2A}(l_{q_1}))$, we estimate the quasi-norm (4). The other required properties for the functions $g_j(x, z)$ are clear by construction. We have

$$|f_j(x - y)| \leqslant f_j^*(x - y) \leqslant c(1 + |2^j y|^a) f_j^*(x).$$

where c is a positive number which is independent of x, y, and j. Using $(F^{-1}\varphi_j)(y) = 2^{jn}(F^{-1}\varphi)(2^j y)$ and the above estimate, (16) then yields

$$\begin{aligned} |g_j(x, z)| &\leqslant c \, \|\{f_k^*\} \mid L_p(l_q)\|^{d(\operatorname{Re} z-\theta)} \, |f_j^*(x)|^{1+b(\operatorname{Re} z-\theta)} \\ &\quad \times \int_{R_n} 2^{jn} |(F^{-1}\varphi)(2^j y)| (1 + |2^j y|^a)^{1+b(\operatorname{Re} z-\theta)} \, dy \\ &\leqslant c' \, \|\{f_k^*\} \mid L_p(l_q)\|^{d(\operatorname{Re} z-\theta)} \, |f_j^*(x)|^{1+b(\operatorname{Re} z-\theta)}. \end{aligned} \tag{17}$$

Here c and c' are independent of j. We have

$$\frac{1}{q_0} - \frac{1}{q} = \theta \left(\frac{1}{q_0} - \frac{1}{q_1} \right) \quad \text{and} \quad \frac{1}{q_1} - \frac{1}{q} = (1 - \theta)\left(\frac{1}{q_1} - \frac{1}{q_0} \right). \tag{18}$$

If we choose $b = q(1/q_1 - 1/q_0)$ then

$$1 - b\theta = \frac{q}{q_0} \quad \text{and} \quad 1 - b(1 - \theta) = \frac{q}{q_1}. \tag{19}$$

If $z = it$ in (17), where t is a real number, then

$$\left(\sum_{j=0}^{\infty} |g_j(x, it)|^{q_0} \right)^{1/q_0} \leqslant c \, \|\{f_k^*\} \mid L_p(l_q)\|^{-d\theta} \left(\sum_{j=0}^{\infty} |f_j^*(x)|^q \right)^{1/q_0}$$

and

$$\|\{g_j(\cdot, it)\} \mid L_{p_0}(l_{q_0})\| \leqslant c \, \|\{f_k^*\} \mid L_p(l_q)\|^{-d\theta} \left(\int_{R_n} \left(\sum_{j=0}^{\infty} |f_j^*(x)|^q \right)^{p_0/q_0} \right)^{1/p_0}. \tag{20}$$

Similarly one obtains that

$$\|g_j(\cdot, 1 - it)| L_{p_1}(l_{q_1})\|$$
$$\leqslant c \,\|\{f_k^*\}, L_p(l_q)\|^{d(1-\theta)} \left(\int_{R_n} \left(\sum_{j=0}^{\infty} f_j^*(x)^q\right)^{p_1/q_1} dx\right)^{1/p_1}. \tag{21}$$

We have $p_0/q_0 = p_1/q_1 = p/q$. If we choose $d = -b$, then (19) yields

$$\frac{p}{p_0} - d\theta = \frac{p}{p_1} - d(1 - \theta) = 1.$$

So by (20) and (21) we have

$$\| g \mid F(L_{p_0}^{2A}(l_{q_0}), L_{p_1}^{2A}(l_{q_1}))\| \leqslant c \,\|\{f_k^*\}, L_p(l_q)\|. \tag{22}$$

By the maximal inequality proved in [9, 2.2.3, formula (15)], the right-hand side of (22) can be estimated from above by $\|\{f_k\} \mid L_p(l_q)\|$. Therefore

$$\| g \mid F(L_{p_0}^{2A}(l_{q_0}), L_{p_1}^{2A}(l_{q_1}))\| \leqslant c \,\| f \mid L_p(l_q)\|, \tag{23}$$

where c is independent of f. Furthermore,

$$g_j(x, \theta) = f_j(x) \qquad \text{if} \quad j = 0, 1, 2, \ldots . \tag{24}$$

Hence, (23), (24), Proposition 2, and the above-mentioned density of the chosen systems f in $L_p{}^A(l_q)$ prove the reversion to (12). The proof is now complete.

Remark 3. On the basis of Peetre's Littlewood–Paley theorem for Hardy spaces mentioned in the Introduction, the complex interpolation theorem for the Hardy spaces proved by Calderón and Torchinsky [3, 11.3] can be reformulated as follows. If $0 < p_0 < \infty$, $0 < p_1 < \infty$, $0 < \theta < 1$, and $1/p = (1 - \theta)/p_0 + \theta/p_1$, then

$$[L_{p_0}^A(l_2), L_{p_1}^A(l_2)]_\theta = L_p{}^A(l_2). \tag{25}$$

The method in [3] is different (although maximal inequalities are used also). Perhaps a combination of the above proof and the Calderón–Torchinsky method yields a proof of Theorem 1 without the additional restriction $p_0/q_0 = p_1/q_1$. Finally we remark that the balls D_k in the definition of $L_p{}^A(l_q)$ can be also replaced by some other sets.

Next we want to prove the so-called interpolation property for linear and bounded mappings. Let T be a linear mapping from $S' \times S' \times \cdots$ into itself, i.e., if $f = \{f_j\}_{j=0}^{\infty} \subset S'$, then $Tf = \{(Tf)_j\}_{j=0}^{\infty} \subset S'$ is a linear operator. T is said to be of convolution type if

$$\operatorname{supp} F[(Tf)_j] \subset \operatorname{supp} Ff_j\,, \qquad j = 0, 1, 2, \ldots . \tag{26}$$

A typical example (which will be of interest later on) is given by

$$(Tf)_j(x) = F^{-1}[\varphi_j Ff_j] = \int_{R_n} (F^{-1}\varphi_j)(x-y) f_j(y)\, dy, \tag{27}$$

where φ_j are given appropriate functions. (We assume that the expressions in (27) are meaningful.) If A is a subset of $S' \times S' \times \cdots$, then we shall say "$T$ is a ...mapping from A into..." instead of the more correct version "the restriction of T to A is a ...mapping from A into...," etc. If T is a linear and bounded operator from a Banach space A into itself, then $\| T \|$ has the usual meaning. The definition of $\| T \|$ works also if A is a quasi-Banach space. In that case $\| T \|$ is a quasi-norm. Finally we say that the above operator T preserves M^{2A} if $Tf \in M^{2A}$ for any system $f \in M^{2A}$ (the definition of M^{2A} precedes Remark 1). Here z appearing in the definition of M^{2A} must be considered as a parameter.

THEOREM 2. *Let $0 < p_0 < \infty, 0 < p_1 < \infty, 0 < q_0 < \infty, 0 < q_1 < \infty$, and $p_0/q_0 = p_1/q_1$. Let T be an M^{2A}-preserving linear and bounded mapping of convolution type from $L_{p_l}^{2A}(l_{q_l})$ into itself, where $l = 0, 1$, with the quasi-norm $\| T \|_l$. If $0 \leqslant \theta \leqslant 1$, then T is a linear and bounded mapping from $L_p{}^A(l_q)$ into itself, where*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \tag{28}$$

with the quasi-norm $\| T \|$, which can be estimated by

$$\| T \| \leqslant c \, \| T \|_0^{1-\theta} \, \| T \|_1^{\theta} .$$

Here c is a positive number which is independent of T.

Proof. If $\theta = 0$ or $\theta = 1$, then the assertion of the theorem is obvious because T is a mapping of convolution type. Let $0 < \theta < 1$. Let $f = \{f_j\}_{j=0}^{\infty} \in L_p{}^A(l_q)$, and let $g = \{ g_j(x, z)\}_{j=0}^{\infty}$ be a corresponding admissible system in the sense of Theorem 1 and the above definition. We want to show that $Tg(x, z) = T\{ g_j(\cdot, z)\}_{j=0}^{\infty}$ is an admissible system for Tf, in particular

$$(Tg)(x, \theta) = Tf. \tag{30}$$

Here z with $0 \leqslant \operatorname{Re} z \leqslant 1$ is a parameter. The counterparts of (4) and (9) follow from the hypotheses. The other required properties are clear because T preserves M^{2A}. Hence, by the above definition

$$Tf \in [L_{p_0}^{A}(l_{q_0}), L_{p_1}^{A}(l_{q_1})]_\theta . \tag{31}$$

We now use Proposition 2. If we restrict the infimum in (10*) to the above system Tg, then

$$\begin{aligned} \| Tf \mid [L_{p_0}^{A}(l_{q_0}), L_{p_1}^{A}(l_{q_1})]_\theta \|^* \\ \leqslant \| T \|_0^{1-\theta} \| T \|_1^{\theta} \inf \| g \mid F(L_{p_0}^{2A}(l_{q_0}), L_{p_1}^{2A}(l_{q_1})) \| \\ \sim \| f \mid [L_{p_0}^{A}(l_{q_0}), L_{p_1}^{A}(l_{q_1})]_\theta \| . \end{aligned}$$

Theorem 1 and Proposition 2 finally yield the desired assertion.

EXAMPLE. T must satisfy two additional conditions: it must preserve M^{2A} and be a mapping of convolution type. Let T be given by (27), where we may restrict ourselves to systems $f = \{f_k\}_{k=0}^{\infty} \subset S'$ satisfying (3). In that case it is easy to see that T preserves M^{2A}, and is a mapping of convolution type if $F^{-1}\varphi_j \in L_1$ for $j = 0, 1, 2, \ldots$. We recall the well-known fact that $\varphi_j \in W_2^{\kappa}$ with $\kappa > n/2$ implies $F^{-1}\varphi_j \in L_1$; cf., e.g., [7, p. 60]. Here W_2^{κ} is the usual Sobolev–Slobodeckij (or Bessel-potential) space. Therefore T satisfies the required additional properties if T is given by (27) and

$$\varphi_j \in W_2^{\kappa}, \quad \kappa > n/2, \qquad \text{where} \qquad j = 0, 1, 2, \ldots . \tag{32}$$

3. FOURIER MULTIPLIERS FOR $L_p^A(l_q)$

First we recall two Fourier multiplier theorems.

(i) If $1 < p < \infty$ and $1 < q < \infty$, then there exists a positive number c such that for all systems $\{f_k\}_{k=0}^{\infty} \subset L_p$ and all systems $\{\varphi_l\}_{l=0}^{\infty}$ one has

$$\begin{aligned} \| \{F^{-1}[\varphi_k F f_k]\} \mid L_p(l_q) \| \\ \leqslant c \, \| \{f_k\} \mid L_p(l_q) \| \sup_{\substack{R>0 \\ 0 \leqslant |\alpha| \leqslant 1+[n/2]}} R^{|\alpha|-n/2} \left(\int_{R/2 \leqslant |\xi| \leqslant 2R} \sum_{l=0}^{\infty} |(D^\alpha \varphi_l)(\xi)|^2 \, d\xi \right)^{1/2} . \end{aligned}$$

Here $\| \cdot \mid L_p(l_q) \|$ has the same meaning as in (2). This is a theorem of the Hörmander type. A proof may be found in [8, pp. 161–165] (in particular Remark 2 on p. 165). Of course, the only systems of interest are $\{f_k\}$ and $\{\varphi_l\}$, for which the two factors on the right-hand side of (33) are finite. Another

Fourier multiplier theorem for $L_p(l_q)$ with $1 < p < \infty$ and $1 < q < \infty$ may be found in [5, p. 241]. Obviously, in the theorem below one can replace the term related to the right-hand side of (33) by the corresponding term in Lizorkin's multiplier theorem.

(ii) If $0 < p < \infty$, $0 < q < \infty$ and $\kappa > n/2 + n/\min(p, q)$ then there exists a positive number c such that for all systems $\{f_k\}_{k=0}^{\infty} \in L_p{}^A(l_q)$ and all systems $\{\varphi_l\}_{l=0}^{\infty}$ of infinitely differentiable functions in R_n

$$\|\{F^{-1}[\varphi_k F f_k]\} \mid L_p(l_q)\| \leqslant c \|\{f_k\} \mid L_p(l_q)\| \sup_{l=0,1,\ldots} \|\varphi_l(2^l \cdot) \mid W_2{}^{\kappa}\|. \tag{34}$$

Here $W_2{}^{\kappa}$ are the usual Sobolev–Slobodeckij or Bessel-potential spaces. A proof may be found in [9, 2.2.3].

THEOREM 3. *Let* $1 < p_0 < \infty$, $1 < q_0 < \infty$, $0 < p_1 < \infty$, $0 < q_1 < \infty$, $p_0/q_0 = p_1/q_1$, *and* $\kappa > n/2 + n/\min(p, q)$. *If* $0 \leqslant \theta \leqslant 1$ *and*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

then there exists a positive number c such that for all systems $\{f_k\}_{k=0}^{\infty} \in L_p{}^A(l_q)$ *and all systems* $\{\varphi_l\}_{l=0}^{\infty}$ *of infinitely differentiable functions in* R_n

$$\begin{aligned} &\|\{F^{-1}[\varphi_k F f_k]\} \mid L_p(l_q)\| \\ &\quad \leqslant c \|\{f_k\} \mid L_p(l_q)\| \times (\sup_l \|\varphi_l(2^l \cdot) \mid W_2{}^{\kappa}\|)^{\theta} \\ &\quad\quad \times \Bigg(\sup_{\substack{R>0 \\ 0 \leqslant |\alpha| \leqslant 1+[n/2]}} R^{|\alpha|-n/2} \Bigg(\int_{R/2 \leqslant |\xi| \leqslant 2R} \sum_{l=0}^{\infty} |(D^{\alpha}\varphi_l)(\xi)|^2 \, d\xi \Bigg)^{1/2} \Bigg)^{1-\theta}. \end{aligned} \tag{35}$$

Proof. We want to apply Theorem 2. First we remark that one can replace $L_p{}^A(l_q)$ in the above multiplier theorem (ii) by $L_p^{2A}(l_q)$. The operator T has the form (27), in particular (26) is satisfied. The example at the end of Section 2 is applicable. Now (35) follows from (29), (33), and (34). The proof is complete.

Remark 4. This theorem is the main goal of the present paper. It looks a little artificial. However, in a later paper we shall apply this theorem to the theory of equivalent quasi-norms for the Hardy–Sobolev spaces as it has been developed in [9, 10]. Then we shall obtain sharp assertions which show that (35) is also sharp in some sense (for details we refer to [11]).

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